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The Non-Compact Quantum Dilogarithm and the Baxter Equations

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A review of the recent formulation of the quantum discrete Liouville model in the strongly coupled regime (corresponding to the Virasoro central charge $1 < c < 25$) is presented. The Q-operator, describing the integrable structure of the model and satisfying a pair of dual Baxter equations, is obtained as a certain non-homogeneous transfer-matrix associated with the six-vertex model.

KEY WORDS: Quantum dilogarithm; quantum integrable systems; Baxter equations.

1. INTRODUCTION

In the paper[8] the quantization problem of the discrete Liouville equation[9] has been considered in the framework of the algebraic approach to quantum integrable models in $1+1$ dimensional discrete space-time developed in refs. 11, 12, 14, 15. It has been shown that the quantum discrete Liouville model in the strongly coupled regime can be formulated as a well defined quantum mechanical problem with unitary evolution operator. The integrable structure of the model has been demonstrated by constructing the Q-operator satisfying a pair of (dual) Baxter equations.[5, 6]

The main purpose of this paper is to review and elaborate on some technical details of the paper.[8] First, we remind briefly the main properties of the noncompact quantum dilogarithm, describe the Q-operator of the quantum discrete Liouville model as well as the Baxter equations. Then, the Q-operator (together with the derivation of the Baxter equations) will be obtained as a specialization of the non-homogeneous transfer-matrix associated with the six-vertex (XXZ) model. The idea of derivation

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of the Baxter equations, given in this paper, originally goes to Baxter himself.[5] In refs. 4, 7 it was realized in the context of the chiral Potts model, and in refs. 1–3 in the context of quantum conformal field theory. The dual Baxter equations in a different context were also considered recently by Smirnov.[13] Our derivation essentially coincides with that presented in ref. 8. It is based on the use of non-ultra-local variables.

2. THE NON-COMPACT QUANTUM DILOGARITHM

In this section, following ref. 8 we review some properties of the non-compact quantum dilogarithm, the main building element for the Q-operator.

Let complex b have a nonzero real part $\Re b \neq 0$. Denote $c_b = (b + b^{-1})/2$. The non-compact QDL, $\psi(z)$, $z \in \mathbb{C}$, $|\Im z| < |\Im c_b|$, is defined by the formula

$$\psi(z) \equiv \exp\left(-\frac{1}{4} \int_{-\infty}^{+\infty} \frac{e^{-2\mathrm{i}zx} dx}{\sinh(xb)\sinh(x/b)\, x} \right) \tag{1}$$

where the singularity at $x = 0$ is put below the contour of integration. This definition implies that $\psi(z)$ is unchanged under substitutions $b \to b^{-1}$, $b \to -b$. Using this symmetry, we choose b to lay in the first quadrant of the complex plane, namely

$$\Re b > 0, \qquad \Im b \geqslant 0$$

which implies that $\Im c_b > 0$.

2.1. Functional Relations

Function (1) satisfies the "inversion" relation

$$\psi(z)\, \psi(-z) = e^{-\mathrm{i}\pi z^2 + \mathrm{i}\pi(1 + 2c_b^c)/6} \tag{2}$$

and a pair of functional equations

$$\psi(z + \mathrm{i}b^{\pm 1}/2) = (1 + e^{2\pi z b^{\pm 1}})\, \psi(z - \mathrm{i}b^{\pm 1}/2) \tag{3}$$

The latter equations enable us to extend definition of the QDL to the entire complex plane.

When b is real or a pure phase, function $\psi(z)$ is *unitary* in the sense that

$$(\psi(z))^* = 1/\psi(z^*), \qquad \text{if either} \quad \Im b = 0, \quad \text{or} \quad |b| = 1 \tag{4}$$

If self-adjoint operators p and q in $L^2(\mathbb{R})$ satisfy the Heisenberg commutation relations

$$[\mathsf{p}, \mathsf{q}] = \frac{1}{2\pi \mathsf{i}} \tag{5}$$

the following operator five term identity holds:

$$\psi(\mathsf{q})\, \psi(\mathsf{p}) = \psi(\mathsf{p})\, \psi(\mathsf{p}+\mathsf{q})\, \psi(\mathsf{q}) \tag{6}$$

For real b this can be proved in the C^*-algebraic framework.[2] See ref. 8 for the proof in the case of complex b by the use of the integral Ramanujan identity.

2.2. Analytic Properties

Let $\Im b^2 > 0$. We can perform the integration in (1) by the residue method. The result can be written in the form

$$\psi(z) = (e^{2\pi(z - c_b)\, b^{-1}};\, \bar{q}^2)_\infty / (e^{2\pi(z + c_b)\, b};\, q^2)_\infty \tag{7}$$

where

$$q = e^{\mathsf{i}\pi b^2}, \qquad \bar{q} = e^{-\mathsf{i}\pi b^{-2}}$$

and

$$(x;\, y)_\infty \equiv \prod_{j=0}^{\infty} (1 - x y^j), \qquad x, y \in \mathbb{C}, \quad |y| < 1$$

Formula (7) defines a meromorphic function on the entire complex plane, satisfying functional equations (2) and (3), with essential singularity at infinity. So, it is the desired extension of definition (1). It is easy to read off location of its poles and zeroes:

$$\text{zeroes of } (\psi(z))^{\pm 1} = \{\pm(c_b + m\mathsf{i}b + n\mathsf{i}b^{-1}) : m, n \in \mathbb{Z}_{\geqslant 0}\}$$

The behavior at infinity depends on the direction along which the limit is taken:

$$\psi(z)\big|_{|z| \to \infty} \approx \begin{cases} 1 & |\arg(z)| > \pi/2 + \arg(b) \\ e^{\mathsf{i}\pi z^2 + \mathsf{i}\pi(1 + 2c_b^2)/6} & |\arg(z)| < \pi/2 - \arg(b) \\ \Theta(\mathsf{i}b^{-1}z;\, -b^{-2})/(\bar{q}^2;\, \bar{q}^2)_\infty & |\arg z - \pi/2| < \arg b \\ (q^2;\, q^2)_\infty / \Theta(\mathsf{i}bz;\, b^2) & |\arg z + \pi/2| < \arg b \end{cases} \tag{8}$$

[2] S. L. Woronowicz: private communication, 1998.

where the standard notation for the Θ-function is used:

$$\Theta(z;\tau)\equiv\sum_{n\in\mathbb{Z}} e^{\mathrm{i}\pi\tau n^2+2\pi\mathrm{i}nz},\qquad \Im\tau>0$$

Thus, for complex b, double quasi-periodic θ-functions, generators of the field of meromorphic functions on complex tori, describe the asymptotic behavior of the non-compact QDL.

2.3. Integral Ramanujan Identity

Consider the following Fourier integral:

$$\Psi(u,v,w)\equiv\int_{\mathbb{R}}\frac{\psi(x+u)}{\psi(x+v)}\,e^{2\pi\mathrm{i}wx}\,dx \tag{9}$$

where

$$\Im(u+c_b)>0,\qquad \Im(-v+c_b)>0,\qquad \Im(u-v)<\Im w<0 \tag{10}$$

Restrictions (10) actually can be considerably relaxed by deforming the integration path in the complex x plane, keeping the asymptotic directions of the two ends within the sectors $\pm(|\arg x|-\pi/2)>\arg b$. So, the enlarged in this way domain for the variables u, v, w has the form:

$$|\arg(\mathrm{i}z)|<\pi-\arg b,\qquad z\in\{w,\,u-v-w,\,v-u-2c_b\} \tag{11}$$

Integral (9) can be evaluated explicitly by the residue method, the result being

$$\Psi(u,v,w)=\frac{\psi(u-v+c_b)\,\psi(-w-c_b)}{\psi(u-v-w+c_b)}\,e^{-2\pi\mathrm{i}w(v-c_b)-\mathrm{i}\pi(1-4c_b^2)/12} \tag{12}$$

$$=\frac{\psi(v+w-u-c_b)}{\psi(v-u-c_b)\,\psi(w+c_b)}\,e^{-2\pi\mathrm{i}w(u+c_b)+\mathrm{i}\pi(1-4c_b^2)/12} \tag{13}$$

where the two expressions in the right hand side are related to each other through the inversion relation (2). In ref. 8 this identity has been demonstrated to be an integral counterpart of the Ramanujan ${}_1\psi_1$ summation formula.

2.4. Fourier Transformation of the QDL

Particular values of $\Psi(u, v, w)$ lead to the following Fourier transformation formulas for the QDL:

$$\phi_+(w) \equiv \int_{\mathbb{R}} \psi(x)\, e^{2\pi i w x}\, dx = \Psi(0, v, w)|_{v \to -\infty}$$

$$= e^{-2\pi i w c_b + i\pi(1-4c_b^2)/12}/\psi(w + c_b) = e^{i\pi w^2 - i\pi(1-4c_b^2)/12}\psi(-w - c_b) \quad (14)$$

and

$$\phi_-(w) \equiv \int_{\mathbb{R}} (\psi(x))^{-1}\, e^{2\pi i w x}\, dx = \Psi(u, 0, w)|_{u \to -\infty}$$

$$= e^{-2\pi i w c_b + i\pi(1-4c_b^2)/12}/\psi(-w - c_b)$$

$$= e^{-i\pi w^2 + i\pi(1-4c_b^2)/12}\psi(w + c_b) \quad (15)$$

The corresponding inverse transformations read:

$$(\psi(x))^{\pm 1} = \int_{\mathbb{R}} \phi_\pm(y)\, e^{-2\pi i x y}\, dy \quad (16)$$

where the pole at $y = 0$ is surrounded from below.

3. QUANTUM DISCRETE LIOUVILLE MODEL

Referring for details to paper,[8] here we briefly describe the essentials of the quantum discrete Liouville equation. It reads as

$$\mathsf{w}_{m,t+1}\mathsf{w}_{m,t-1} = (1 + q\mathsf{w}_{m+1,t})(1 + q\mathsf{w}_{m-1,t}) \quad (17)$$

where

$$m, t \in \mathbb{Z}, \qquad m + t = 0 \pmod 2$$

dynamical variables $\mathsf{w}_{m,t}$ are elements of the observable algebra (see below), and $q = \exp(i\pi b^2)$, b is the *coupling* constant (or square root thereof).

The Virasoro central charge in the continuous limit is expected to be given in terms of b by the standard formula

$$c = 1 + 6(b + b^{-1})^2$$

3.1. Algebra of Observables

The algebra of observables is generated by a finite set of self-adjoint operators $\{\mathsf{f}_m, m \in \mathbb{Z}\}$, satisfying the periodicity condition with period $2N$,

$$\mathsf{f}_{m+2N} = \mathsf{f}_m, \qquad N > 1, \quad N = 1 \pmod 2 \tag{18}$$

and the defining commutation relations:

$$[\mathsf{f}_m, \mathsf{f}_n] = \begin{cases} (-1)^m (2\pi \mathsf{i})^{-1}, & \text{if} \quad n = m \pm 1 \\ 0, & \text{otherwise} \end{cases} \tag{19}$$

The initial data for the field variables in (17) are just the exponentials of the generating elements:[3]

$$\mathsf{w}_{m,\, -[m+1]_2} = e^{2\pi b \mathsf{f}_m}$$

where

$$[m]_2 \equiv (1 - (-1)^m)/2, \qquad \forall m \in \mathbb{Z} \tag{20}$$

The evolution operator U, satisfying $\mathsf{U}^{-1} \mathsf{w}_{m,m} \mathsf{U} = \mathsf{w}_{m,n+2}$, has the form

$$\mathsf{U} = \prod_{j=1}^{N} \psi(\mathsf{f}_{2j})\, \mathsf{F} \prod_{k=1}^{N} \psi(\mathsf{f}_{2k-1}) \tag{21}$$

where operator F changes signs of the generators:

$$\mathsf{F} \mathsf{f}_m = -\mathsf{f}_m \mathsf{F}$$

4. BAXTER EQUATIONS

The order in products with non-commuting entries will be indicated as follows:

$$\prod_{n \geqslant i \geqslant m} a_i \equiv a_n a_{n-1} \cdots a_{m+1} a_m, \qquad \prod_{m \leqslant i \leqslant n} a_i \equiv a_m a_{m+1} \cdots a_{n-1} a_n$$

Consider algebra $\mathscr{B}_{2N}$ of operators with linear basis of the form

$$\prod_{1 \leqslant i \leqslant 2N} e^{2\pi \mathsf{i} \mathsf{f}_i x_i}$$

[3] Because of the duality symmetry $b \leftrightarrow b^{-1}$ in the theory, there actually exist two types of exponential fields $\exp(2\pi b^{\pm 1} f_m)$ which satisfy two quantum Liouville equations.

where self-adjoint operators f_i satisfy commutation relations (19), and variables x_i take real or complex values.

The ascending *cyclic product* is a set of linear mappings,

$$\mathsf{o}_j^+ : \mathscr{B}_{2N} \to \mathscr{B}_{2N}, \qquad j \in \mathbb{Z}, \quad \mathsf{o}_j^+ = \mathsf{o}_{j+2N}^+$$

acting diagonally on basis monomials:

$$\mathsf{o}_1^+ \left(\prod_{1 \leqslant i \leqslant 2N} e^{2\pi \mathsf{i} \mathsf{f}_i x_i} \right) \equiv e^{2\pi \mathsf{i} x_{2N} x_1} \prod_{1 \leqslant i \leqslant 2N} e^{2\pi \mathsf{i} \mathsf{f}_i x_i} \equiv \mathsf{o}_j^+ \left(\prod_{j \leqslant i \leqslant j+2N-1} e^{2\pi \mathsf{i} \mathsf{f}_i x_i} \right)$$

Consider the transfer-matrices

$$\mathsf{t}^{\pm}(\mu) = \mathsf{o}_1^+ \operatorname{Tr} \prod_{1 \leqslant j \leqslant 2N} \mathsf{L}_j^{\pm} \tag{22}$$

where

$$\mathsf{L}_j^{\pm} \equiv \begin{pmatrix} e^{-(-1)^j \pi b^{\pm 1} \mathsf{f}_j} & 0 \\ 0 & e^{(-1)^j \pi b^{\pm 1} \mathsf{f}_j} \end{pmatrix} \begin{pmatrix} e^{(-1)^j \pi b^{\pm 1} \mu} & e^{-(-1)^j \pi b^{\pm 1} \mu} [j+1]_2 \\ e^{-(-1)^j \pi b^{\pm 1} \mu} & e^{(-1)^j \pi b^{\pm 1} \mu} \end{pmatrix} \tag{23}$$

see also (20), and the trace is that of two-by-two matrices, and

$$\mathsf{Q}(\mu) = \mathsf{o}_1^+ \left(\prod_{1 \leqslant j \leqslant 2N} w_{[j]_2}(\mu, \mathsf{f}_j) \right) \mathsf{G} \tag{24}$$

where

$$e^{-2\pi \mathsf{i} \mu x} \psi(x-\mu)\, w_i(\mu, x) \equiv \begin{cases} \psi(x+\mu), & \text{if} \quad i=0 \\ 1, & \text{if} \quad i=1 \end{cases} \tag{25}$$

and G is defined by its action on the generators:

$$\mathsf{G} \mathsf{f}_m = (-1)^m \, \mathsf{f}_{m-1} \mathsf{G} \tag{26}$$

These transfer-matrices commute among themselves

$$[\mathsf{t}^{\varepsilon}(\mu), \mathsf{t}^{\pm}(\nu)] = [\mathsf{Q}(\mu), \mathsf{Q}(\nu)] = [\mathsf{t}^{\pm}(\mu), \mathsf{Q}(\nu)] = 0, \qquad \varepsilon = \pm$$

and solve the following Baxter equations:

$$\mathsf{t}^{\pm}(\mu)\, \mathsf{Q}(\mu) = \mathsf{Q}(\mu + \mathsf{i} b^{\pm 1}/2) + (1 - e^{-4\pi b^{\pm 1}} \mu)^N \, \mathsf{Q}(\mu - \mathsf{i} b^{\pm 1}/2) \tag{27}$$

A remark is in order. The product of two neighboring L-operators $\mathsf{L}^+_{2i}\mathsf{L}^+_{2i+1}$ is equivalent to the spectral parameter dependent L-operator introduced in ref. 10 for the description of the Liouville equation in the framework of inverse scattering method.

4.1. Connection to the Evolution Operator

The evolution operator for the lattice Liouville equation (17) is related to the Baxter operator through the following formula:

$$\mathsf{U}^{-1} = (\mathsf{Q}(0))^2\,\mathsf{P} \tag{28}$$

where P and F act on generators according to formulae

$$\mathsf{F}\mathsf{f}_m = -\mathsf{f}_m\mathsf{F}, \qquad \mathsf{P}\mathsf{f}_m = \mathsf{f}_{m+2}\mathsf{P}$$

and are given by integer powers of the operator G:

$$\mathsf{F} = \mathsf{G}^{2N}, \qquad \mathsf{P} = \mathsf{G}^{-2N-2}$$

The Q-operator (24) commutes with operator G. That means $\mathsf{Q}(\mu)$ is the generating function of conserved quantities for the quantum discrete Liouville equation:

$$[\mathsf{U}, \mathsf{Q}(\mu)] = 0$$

Moreover, it is clear that the spectrum of the evolution operator is straightforwardly determined from the spectrum of the Q-operator.

5. DERIVATION OF THE BAXTER EQUATIONS

For any positive integer M we define algebra $\mathscr{A}_M$ with linear basis of the form $\prod_{M\geqslant i\geqslant 1} e^{2\pi \mathrm{i}\mathsf{a}_i x_i}$, where self-adjoint operators $\{\mathsf{a}_i\}_{i\in\mathbb{Z}}$, $\mathsf{a}_{i+M}=\mathsf{a}_i$, satisfy the "lattice current algebra" relations:

$$[\mathsf{a}_m, \mathsf{a}_n] = \begin{cases} (2\pi\mathrm{i})^{-1}, & \text{if} \quad n = m+1 \\ 0, & \text{if} \quad |m-n| \neq 1 \end{cases}$$

and variables x_i take real or complex values.

Define the descending cyclic product as a set of invertible linear mappings

$$\mathsf{o}_j^- : \mathscr{A}_M \to \mathscr{A}_M, \qquad j \in \mathbb{Z}, \quad \mathsf{o}_j^- = \mathsf{o}_{j+M}^-$$

diagonally acting on basis monomials:

$$\mathsf{o}_1^- \left(\prod_{M \geqslant i \geqslant 1} e^{2\pi \mathrm{i} \mathsf{a}_i x_i} \right) \equiv e^{-2\pi \mathrm{i} x_1 x_M} \prod_{M \geqslant i \geqslant 1} e^{2\pi \mathrm{i} \mathsf{a}_i x_i} \equiv \mathsf{o}_j^- \left(\prod_{j+M-1 \geqslant i \geqslant j} e^{2\pi \mathrm{i} \mathsf{a}_i x_i} \right)$$

The usual product of two cyclic products can be written as a single cyclic product

$$\mathsf{o}_1^- \left(\prod_{M \geqslant i \geqslant 1} e^{2\pi \mathrm{i} \mathsf{a}_i x_i} \right) \mathsf{o}_1^- \left(\prod_{M \geqslant j \geqslant 1} e^{2\pi \mathrm{i} \mathsf{a}_j y_j} \right)$$

$$= e^{2\pi \mathrm{i}(x_1 y_M + x_M y_1)} \mathsf{o}_1^- \left(\left(\prod_{M \geqslant i \geqslant 1} e^{2\pi \mathrm{i} \mathsf{a}_i x_i} \right) \prod_{M \geqslant j \geqslant 1} e^{2\pi \mathrm{i} \mathsf{a}_j y_j} \right) \tag{29}$$

$$= e^{2\pi \mathrm{i} x_M y_1} \mathsf{o}_1^- \left(\left(\prod_{M \geqslant i \geqslant 2} e^{2\pi \mathrm{i} \mathsf{a}_i x_i} \right) e^{2\pi \mathrm{i} \mathsf{a}_M y_M} e^{2\pi \mathrm{i} \mathsf{a}_1 x_1} \prod_{M-1 \geqslant j \geqslant 1} e^{2\pi \mathrm{i} \mathsf{a}_j y_j} \right) \tag{30}$$

$$= e^{2\pi \mathrm{i} x_M y_1} \mathsf{o}_1^- \left(e^{2\pi \mathrm{i} \mathsf{a}_M x_M} \left(\prod_{M \geqslant i \geqslant 2} e^{2\pi \mathrm{i} \mathsf{a}_{i-1} x_{i-1}} e^{2\pi \mathrm{i} \mathsf{a}_i y_i} \right) e^{2\pi \mathrm{i} \mathsf{a}_1 y_1} \right) \tag{31}$$

First, we shall derive the Baxter equations in the most general form, and then identify the case, corresponding to the quantum discrete Liouville model, as a particular (limiting) case.

Define completely non-homogeneous transfer-matrices

$$\mathscr{T}^{\pm}(\mu) = \mathsf{o}_1^- \operatorname{Tr} \prod_{M \geqslant j \geqslant 1} \Lambda_j^{\pm}$$

$$\Lambda_j^{\pm} = \begin{pmatrix} e^{-\pi b^{\pm 1} \mathsf{a}_j} & 0 \\ 0 & e^{\pi b^{\pm 1} \mathsf{a}_j} \end{pmatrix} \begin{pmatrix} e^{\pi b^{\pm 1}(2\alpha_j + \mu)} & e^{\pi b^{\pm 1}(2\beta_j - \mu)} \\ e^{\pi b^{\pm 1}(2\gamma_j - \mu)} & e^{\pi b^{\pm 1}(2\delta_j + \mu)} \end{pmatrix} \tag{32}$$

where μ, α_j, β_j, γ_j, δ_j, $1 \leqslant j \leqslant M$, are complex parameters, and

$$\mathscr{Q}(\mu) = \mathsf{o}_1^- \left(\prod_{M \geqslant j \geqslant 1} W_j(\mu, \mathsf{a}_j) \right) \mathsf{P}_{\mathscr{A}_M}$$

$$W_j(\mu, x) \equiv \frac{\psi(x + \mu + \delta_j - \beta_j)}{\psi(x - \mu + \gamma_j - \alpha_j)} e^{2\pi \mathrm{i} x(\mu + \alpha_j - \beta_j)} \tag{33}$$

and $\mathsf{P}_{\mathscr{A}_M}$ is the translation operator in the algebra $\mathscr{A}_M$:

$$\mathsf{P}_{\mathscr{A}_M} \mathsf{a}_j = \mathsf{a}_{j+1} \mathsf{P}_{\mathscr{A}_M}, \qquad \mathsf{P}_{\mathscr{A}_M}^M = 1$$

Lemma 1. The following Baxter equations are satisfied:

$$\mathcal{T}^{\pm}(\mu)\,\mathcal{Q}(\mu) = \mathcal{Q}(\mu + \mathrm{i}b^{\pm 1}/2) \prod_{j=1}^{M} e^{\pi b^{\pm 1}(\alpha_j + \beta_j)} + \mathcal{Q}(\mu - \mathrm{i}b^{\pm 1}/2) \prod_{j=1}^{M} 2\sinh(\pi b^{\pm 1}(2\mu + \sigma_j))\, e^{\pi b^{\pm 1}(\gamma_j + \delta_j)} \tag{34}$$

where $\sigma_j \equiv \alpha_j + \delta_j - \beta_j - \gamma_j$.

Proof. To begin with, note that the following equations hold:

$$\Lambda^{\varepsilon}_{i-1}\, W_i(\mu, \mathsf{a}_i) = \mathsf{R}^{\varepsilon}_i \Lambda^{\varepsilon}_{i-1}, \qquad \varepsilon = \pm \tag{35}$$

where

$$\mathsf{R}^{\varepsilon}_i = \begin{pmatrix} W_i(\mu, \mathsf{a}_i + \mathrm{i}b_\varepsilon/2) & 0 \\ 0 & W_i(\mu, \mathsf{a}_i - \mathrm{i}b_\varepsilon/2) \end{pmatrix}, \qquad b_{\pm} \equiv b^{\pm 1}$$

Next, introducing expansions

$$\Lambda^{\varepsilon}_i = \sum_{x = \pm \mathrm{i}b_\varepsilon/2} \mathsf{l}^{\varepsilon}_i(x)\, e^{2\pi \mathrm{i} x \mathsf{a}_i} \tag{36}$$

we calculate

$$\mathcal{T}^{\varepsilon}(\mu)\,\mathcal{Q}(\mu)\,\mathsf{P}^{-1}_{\mathscr{A}_M}$$

$$= \mathsf{o}_1^{-}\,\mathrm{Tr} \sum_{x = \pm \mathrm{i}b_\varepsilon/2} e^{2\pi \mathrm{i} x \mathsf{a}_M} \mathsf{l}^{\varepsilon}_M(x) \left(\prod_{M \geqslant i \geqslant 2} \Lambda^{\varepsilon}_{i-1} W_i(\mu, \mathsf{a}_i) \right) W_1(\mu, \mathsf{a}_1 + x)$$

—we have used formula (31) for the product of two cyclic products—

$$= \mathsf{o}_1^{-}\,\mathrm{Tr} \sum_{x = \pm \mathrm{i}b_\varepsilon/2} e^{2\pi \mathrm{i} x \mathsf{a}_M} \mathsf{l}^{\varepsilon}_M(x)\, \mathsf{R}^{\varepsilon}_M \left(\prod_{M-1 \geqslant i \geqslant 2} \Lambda^{\varepsilon}_i \mathsf{R}^{\varepsilon}_i \right) \Lambda^{\varepsilon}_1 W_1(\mu, \mathsf{a}_1 + x)$$

—Eq. (35) has been applied $M-2$ times—

$$= \mathsf{o}_2^{-}\,\mathrm{Tr} \sum_{x = \pm \mathrm{i}b_\varepsilon/2} \Lambda^{\varepsilon}_1 W_1(\mu, \mathsf{a}_1 + x)\, e^{2\pi \mathrm{i} x \mathsf{a}_M} \mathsf{l}^{\varepsilon}_M(x)\, \mathsf{R}^{\varepsilon}_M \prod_{M-1 \geqslant i \geqslant 2} \Lambda^{\varepsilon}_i \mathsf{R}^{\varepsilon}_i$$

—we have moved operator $\Lambda^{\varepsilon}_i W_1(\mu, \mathsf{a}_1 + x)$ to the leftmost position by using simultaneously cyclic properties of the trace and the cyclic product—

$$= \mathsf{o}_2^{-}\,\mathrm{Tr} \sum_{x = \pm \mathrm{i}b_\varepsilon/2} \Lambda^{\varepsilon}_1 e^{2\pi \mathrm{i} x \mathsf{a}_M} \mathsf{l}^{\varepsilon}_M(x)\, W_1(\mu, \mathsf{a}_1)\, \mathsf{R}^{\varepsilon}_M \prod_{M-1 \geqslant i \geqslant 2} \Lambda^{\varepsilon}_i \mathsf{R}^{\varepsilon}_i$$

—operator $W_1(\mu, \mathsf{a}_1)$ has changed its position by two steps to the right—

$$= \mathsf{o}_2^- \operatorname{Tr}\left(\Lambda_1^\varepsilon \Lambda_M^\varepsilon W_1(\mu, \mathsf{a}_1)\, \mathsf{R}_M^\varepsilon \prod_{M-1 \geqslant i \geqslant 2} \Lambda_i^\varepsilon \mathsf{R}_i^\varepsilon \right)$$

—now we have taken off the expansion (36)—

$$= \mathsf{o}_1^- \operatorname{Tr} \prod_{M \geqslant i \geqslant 1} \Lambda_i^\varepsilon \mathsf{R}_i^\varepsilon = \mathsf{o}_1^- \operatorname{Tr} \prod_{M \geqslant i \geqslant 1} \mathscr{L}_i^\varepsilon$$

we have applied one more time Eq. (35), brought operator $\Lambda_i^\varepsilon \mathsf{R}_1^\varepsilon$ back to the rightmost position, and made the gauge transformation of the form

$$\mathscr{L}_i^\varepsilon = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \Lambda_i^\varepsilon \mathsf{R}_i^\varepsilon \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \Delta_i^- W_i(\mu - \mathrm{i}b_\varepsilon/2, \mathsf{a}_i) & W_i(\mu, \mathsf{a}_i - \mathrm{i}b_\varepsilon/2)\, e^{-\pi b_\varepsilon(\mathsf{a}_i + \mu - 2\beta_i)} \\ 0 & \Delta_i^+ W_i(\mu + \mathrm{i}b_\varepsilon/2, \mathsf{a}_i) \end{pmatrix}$$

$$\Delta_i^- \equiv 2\sinh(\pi b_\varepsilon(2\mu + \sigma_i))\, e^{\pi b_\varepsilon(\gamma_i + \delta_i)}, \qquad \Delta_i^+ \equiv e^{\pi b_\varepsilon(\alpha_i + \beta_i)}$$

The right hand side of the Baxter relations now follows easily. ∎

Let us choose now $M = 3N$. Define two sets of parameters

$$\mathscr{X} \equiv \{\alpha_{3j-1}, \delta_{3j-1}, \delta_{3j-2} \mid 1 \leqslant j \leqslant N\}$$

and

$$\mathscr{Y} \equiv \{\alpha_i, \beta_i, \gamma_i, \delta_i \mid 1 \leqslant i \leqslant M\} \setminus \mathscr{X}$$

For any complex a and any set of complex numbers S we shall denote by $S \to a$ the limit where each element of S approaches a.

Lemma 2. Operators $\mathscr{T}^\pm(\mu)$ and

$$\mathscr{Q}^{\mathrm{ren}}(\mu) \equiv \mathscr{Q}(\mu) \prod_{j=1}^{N} e^{-\mathrm{i}\pi(\mu + \alpha_{3j-1} - \gamma_{3j-1})^2} \tag{33}$$

are nonsingular in the limit, when $\mathscr{X} \to -\infty$ and $\mathscr{Y} \to 0$, and the corresponding limiting operators, $\mathscr{T}_{\lim}^\pm(\mu)$ and $\mathscr{Q}_{\lim}^{\mathrm{ren}}(\mu)$, satisfy the following Baxter equations:

$$e^{\pi b^{\pm 1} N(\mu + \mathrm{i}b^{\pm 1}/4)} \mathscr{T}_{\lim}^\pm(\mu)\, \mathscr{Q}_{\lim}^{\mathrm{ren}}(\mu)$$
$$= \mathscr{Q}_{\lim}^{\mathrm{ren}}(\mu + \mathrm{i}b^{\pm 1}/2) + (1 - e^{-4\pi b^{\pm 1}\mu})^N\, \mathscr{Q}_{\lim}^{\mathrm{ren}}(\mu - \mathrm{i}b^{\pm 1}/2) \tag{37}$$

Proof. The non-singularity of the limit is easily seen from the definitions (32) and (33) of the operators, and the behavior of the QDL at infinity described by Eq. (8). Baxter equations (37) for the limiting operators are just the limits of Eq. (34). ▮

Define a faithful, star-structure preserving homomorphism of algebras

$$\kappa\colon \mathscr{B}_{2N} \to \mathscr{A}_{3N}$$

by its action on the generators:

$$\mathsf{f}_{2i} \overset{\kappa}{\mapsto} \mathsf{a}_{-3i}, \qquad \mathsf{f}_{2i+1} \overset{\kappa}{\mapsto} \mathsf{a}_{-3i-2} - \mathsf{a}_{-3i-1} \tag{38}$$

Additionally, define the image of the flip-shift operator G (see Eq. (26)) by the formula:

$$\mathsf{G} \overset{\kappa}{\mapsto} \left(\prod_{j=1}^{N} e^{\mathrm{i}\pi \mathsf{a}_{3j-1}^2} \right) \mathsf{P}_{\mathscr{A}_{3N}} \tag{39}$$

Lemma 3. Mapping κ is such that

$$\mathsf{t}^{\pm}(\mu) \overset{\kappa}{\mapsto} e^{\pi b^{\pm 1} N(\mu + \mathrm{i} b^{\pm 1}/4)} \mathscr{T}^{\pm}_{\lim}(\mu) \tag{40}$$

$$\mathsf{Q}(\mu) \overset{\kappa}{\mapsto} e^{\mathrm{i}\pi N(1+2c_b^2)/6} \mathscr{Q}^{\mathrm{ren}}_{\lim}(\mu) \tag{41}$$

where $\mathsf{Q}(\mu)$ and $\mathsf{t}^{\pm}(\mu)$ are defined by Eqs. (22)–(25).

Proof. Let $\varepsilon = \pm$. First, we have compatible cyclic products:

$$\kappa \circ \mathsf{o}^{+}_{2i}|_{\mathscr{B}_{2N}} = \mathsf{o}^{-}_{1-3i}|_{\mathscr{A}_{3N}} \circ \kappa, \qquad \kappa \circ \mathsf{o}^{+}_{2i+1}|_{\mathscr{B}_{2N}} = \mathsf{o}^{-}_{-3i}|_{\mathscr{A}_{2N}} \circ \kappa \tag{42}$$

Next, it is easily seen that

$$\lim_{\mathscr{X} \to -\infty,\, \mathscr{Y} \to 0} \Lambda^{\varepsilon}_{3i} = \kappa(\mathsf{L}^{\varepsilon}_{-2i})$$

$$e^{\pi b_\varepsilon(\mu + \mathrm{i} b_\varepsilon/4)} \lim_{\mathscr{X} \to -\infty,\, \mathscr{Y} \to 0} \Lambda^{\varepsilon}_{3i-1} \Lambda^{\varepsilon}_{3i-2} = \kappa(\mathsf{L}^{\varepsilon}_{-2i+1})$$

This proves Eq. (40). Finally, we have

$$e^{\mathrm{i}\pi N(1+2c_b^2)/6} \mathscr{Q}^{\mathrm{ren}}_{\lim}(\mu) = \mathsf{o}^{-}_{1} \left(\prod_{N \geqslant j \geqslant 1} w_0(\mu, \mathsf{a}_{3j})\, e^{\mathrm{i}\pi \mathsf{a}_{3j-1}^2} w_1(\mu, \mathsf{a}_{3j-2}) \right) \mathsf{P}_{\mathscr{A}_{3N}}$$

—we have used notation (25)—

$$= \mathsf{o}^{-}_{1} \left(\prod_{N \geqslant 1 \geqslant 1} w_0(\mu, \mathsf{a}_{3j})\, w_1(\mu, \mathsf{a}_{3j-2} - \mathsf{a}_{3j-1}) \right) \left(\prod_{j=1}^{N} e^{\mathrm{i}\pi \mathsf{a}_{3j-1}^2} \right) \mathsf{P}_{\mathscr{A}_{3N}}$$

—we have pulled all the Gaussian exponentials out of the cyclic product—

$$=\mathsf{o}_0^- \circ \kappa\left(\prod_{1\leqslant j\leqslant 2N} w_{[j]_2}(\mu, \mathsf{f}_j)\right)\kappa(\mathsf{G})=\kappa(\mathsf{Q}(\mu))$$

Here we have replaced operator $w_0(\mu, \mathsf{a}_{3N})$ from the leftmost to the rightmost position (simultaneously replacing o_1^- by o_0^-), then used definitions (38) and (39), and applied the second formula in Eqs. (42) at $i=0$. ∎

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